Monotone approach to the Moreau-Yosida regularization

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We extend the classical Moreau-Yosida regularization to include potentials of monotone operators not necessarily being duality mappings. Let X be a normed space. Let $A\colon X \to X^*$ be a potential operator with the potential $F\colon X \to \mathbb{R}$. We define for $\varepsilon > 0$ the A-Moreau-Yosida regularization as follows

$$f_\varepsilon^A(u) = \inf_{v\in X}\left\{\frac{1}{\varepsilon}F(u-v) + f(v)\right\}, \qquad u \in X.$$

The following assertions hold:

a) $f_\varepsilon^A(u) = \frac{1}{\varepsilon}F(u-u_\varepsilon) + f(u_\varepsilon)$ for every $u \in X$,

b) $f\left(u_{\varepsilon_1}\right) \le f_{\varepsilon_1}^A(u) \le f_{\varepsilon_2}^A(u) \le f(u)$ for $u \in X$ and $0 < \varepsilon_2 < \varepsilon_1$,

c) $u_\varepsilon \to u$ as $\varepsilon \to 0^+$ for $u \in \operatorname{dom}(f)$,

d) $\lim\limits_{\varepsilon\to 0^+} f_\varepsilon^A(u) = f(u)$ for every $u \in X$.

e) For $u \in \operatorname{dom}(\partial f)$ there holds $A_\varepsilon(u) \rightharpoonup A_0(u) \in \partial f(u)$ as $\varepsilon \to 0^+$, where

$$A_0(u) := \operatorname*{argmin}_{\xi\in\partial f(u)}\{\|\xi\|\}.$$

If X^* is uniformly convex, then $A_\varepsilon(u) \to A_0(u)$ as $\varepsilon \to 0^+$.

f) The mapping $\varepsilon \mapsto f_\varepsilon^A(u)$ is differentiable a.e. on $(0,\infty)$ with

$$\frac{\mathrm{d}f_\varepsilon^A(u)}{\mathrm{d}\varepsilon} = -\frac{1}{\varepsilon}F(u-u_\varepsilon).$$

g) If X is uniformly convex and A is continuous, then A_ε is continuous.

The Mosco convergence of such regularisation is also investigated.

References

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